# SCHUR FUNCTIONS

## 1. Schur family

A Schur symmetric function, $S$-function, $\{\lambda\}$, indexed by a partition $\lambda$ is the sum of all monomials given by SSYT's of shape $\lambda$, with entries from 1 to $d$. An expansion of $\{\lambda\}$ in terms of monomial symmetric functions is given by

$$\{\lambda\} = \sum_{\mu \vdash |\lambda|} K_{\lambda\mu}\, \mathrm{m}_\mu$$

with the coefficients *Kostka numbers*. $K_{\lambda\mu}$ counts the number of SSYT's of shape $\lambda$ that contribute the monomial $\mathrm{m}_\mu$ to the sum.

**Example.** For example, $\lambda = [n]$ gives the diagram consisting of a single row of length $n$, yielding the homogeneous symmetric function $\mathrm{h}_n$. Similarly, the partition $[1^n]$ corresponds to a diagram consisting of a single column of length $n$, thus yielding the elementary symmetric function $\mathrm{e}_n$.

**Example.** In three variables, the $S$-functions for $n = 3$ are

$$\begin{aligned}
\{3\} &= \mathrm{h}_3 \\
\{21\} &= \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array} \\
&\quad + \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 2&3\\\hline 3\\\cline{1-1}\end{array} \\
&= 2\,\mathrm{m}_{(111)} + \mathrm{m}_{(21)} \\
\{111\} &= \mathrm{e}_3
\end{aligned}$$

Taken together the set of $S$-functions $\{\lambda\}$ ranging over appropriate partitions $\lambda$ form a basis for the symmetric functions under consideration.

The $S$-functions have expansions in terms of the $\mathrm{p}_\lambda$'s. The coefficients involve the character table of the symmetric group $(\chi^\lambda_\rho)$.

**Proposition 1.1.** *For $\lambda \vdash n$, we have the expansion*

$$\{\lambda\} = \sum_{\rho \vdash n} \chi^\lambda_\rho \frac{\mathrm{p}^\rho}{z_\rho}$$

**Example.** The partitions of 3 are $[111], [21], [3]$. We look up the character table of $S_3$:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

with rows and columns labelled by the partitions of 3. We have values of $z_\rho$ for 3: (3,2,6). Recovering the expansions for $\mathrm{e}_3$ and $\mathrm{h}_3$, we find a new one:

$$\begin{aligned} \mathrm{h}_3 &= \mathrm{p}_1^3/6 + \mathrm{p}_2\mathrm{p}_1/2 + \mathrm{p}_3/3 \\ \{21\} &= \mathrm{p}_1^3/3 - \mathrm{p}_3/3 \\ \mathrm{e}_3 &= \mathrm{p}_1^3/6 - \mathrm{p}_2\mathrm{p}_1/2 + \mathrm{p}_3/3 \end{aligned}$$