MONOMIAL SYMMETRIC FUNCTIONS

1. Monomial family

Monomial symmetric functions are denoted m_λ indexed by partitions λ. (Note for subscripts we drop the [ ] notation for partitions, optionally using parens.)

Given $\lambda = [4, 3, 3]$, we form $\xi_{(433)} = x_1^4 x_2^3 x_3^3$.

This corresponds to the Young diagram

1	1	1	1
2	2	2	
3	3	3	

Then

$$\mathrm{m}_{(433)} = \text{the minimal symmetric polynomial in } d \text{ variables containing } \xi_{(433)}$$

For $d = 3$, we get

$$\mathrm{m}_{(433)} = x_1^4 x_2^3 x_3^3 + x_1^3 x_2^4 x_3^3 + x_1^3 x_2^3 x_3^4$$

noticing that the exponents are permuted, not the subscripts.

In general, given λ, form the monomial $\xi_\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_L^{\lambda_L}$ and symmetrize over the exponents. This yields all terms containing the variables $\{x_1, \ldots, x_L\}$.

Example. Consider $L = 1$, $\lambda = [n]$. Start with x_1^n and there is no symmetrization with respect to the exponent(s). So pick each variable in turn and add in x_i^n at each step. We get

$$\mathrm{m}_{(n)} = \mathrm{p}_n = x_1^n + \cdots + x_d^n$$

the n^{th} power sum function.

Example. On the other hand, if $\lambda = [111]$, we start with $x_1 x_2 x_3$, again no symmetrization with respect to the exponents. To symmetrize over the variables, we add up the corresponding products of 3 variables at a time. Thus, we get the elementary symmetric function e_3. In general, with $\rho(\lambda) = (1^n)$, i.e., all 1's, we get

$$\mathrm{m}_{(1^n)} = \mathrm{e}_n \ .$$

Example. Observe that each monomial in m_λ is of homogeneous degree $|\lambda| = n$, say. And each λ produces a different monomial function. So the sum over all m_λ with $\lambda \vdash n$ is the sum over all monomials of homogeneous degree equal to n which is the n^{th} homogeneous symmetric function. That is,

$$\mathrm{h}_n = \sum_{\lambda \vdash n} \mathrm{m}_\lambda \ .$$

Proposition 1.1. *The number of terms in* m_λ*, with* $\rho(\lambda) = (1^{\rho_1} 2^{\rho_2} \cdots n^{\rho_n})$*, is*

$$\#\mathrm{m}_\lambda = \binom{d}{L} \frac{L!}{\rho_1!\, \rho_2! \cdots \rho_n!}$$

Proof. Think of constructing m_λ iterating two steps. First pick a subset of L variables, $\binom{d}{L}$ ways. Apply the exponents and symmetrize over the exponents. Repeat for each L-subset and sum everything up. Beginning with a monomial of the form

$$x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_L}^{\lambda_L} = \text{product with exponents } \rho_1\ 1's,\ \rho_2\ 2's,\ \text{etc.}$$

symmetrizing over the exponents provides a multinomial factor of

$$\frac{L!}{\rho_1!\, \rho_2! \cdots \rho_n!}$$

as required. □

Example. For $\lambda = [433]$, we get, for $d = 3$,

$$\#\mathrm{m}_{(3^2 4^1)} = \binom{3}{3} \frac{3!}{2!\, 1!} = 3$$

as seen above.

Remark. Note that the number of variables, d must satisfy $d \geq L$. In other words, $\mathrm{m}_\lambda = 0$ if $L > d$.

The monomial functions m_λ comprise a linear basis for the corresponding symmetric functions of a given number of variables.