HOMOGENEOUS SYMMETRIC FUNCTIONS

1. Homogeneous family

A homogeneous symmetric function with a single integer index, n, is the sum of all monomials of homogeneous degree n. That is,

$$x_1^{n_1} x_2^{n_2} \cdots x_d^{n_d}$$

has homogeneous degree n if the total degree $n_1 + \cdots + n_d = n$. In terms of monomial symmetric functions,

$$\mathrm{h}_n = \sum_{\lambda \vdash n} \mathrm{m}_\lambda$$

Example. Let $d = 3$. We have

$$\begin{aligned} \mathrm{h}_4 &= x_1^4 + x_2^4 + x_3^4 + x_1^3 x_2 + x_1^3 x_3 + x_1 x_2^3 + x_1 x_3^3 + x_2^3 x_3 + x_2 x_3^3 \\ &\qquad + x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2 + x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2 \\ &= \mathrm{m}_{(4)} + \mathrm{m}_{(31)} + \mathrm{m}_{(22)} + \mathrm{m}_{(211)} \end{aligned}$$

with $\mathrm{m}_{(1111)} = 0$ as $L([1111]) > 3$.

The homogeneous symmetric function indexed by λ is the product of the corresponding single-indexed functions:

$$\mathrm{h}_\lambda = \mathrm{h}_{\lambda_1} \mathrm{h}_{\lambda_2} \cdots \mathrm{h}_{\lambda_L} = \mathrm{h}_1^{\rho_1} \mathrm{h}_2^{\rho_2} \cdots \mathrm{h}_n^{\rho_n} = \mathrm{h}^\rho$$

in multi-index notation. Taken together $\{\mathrm{h}_\lambda\}_\lambda$ form a basis for the symmetric functions under consideration.
We have the expansion in terms of monomial symmetric functions.

Proposition 1.1. *For given partitions λ, μ, let $S_{\lambda\mu}$ denote the number of nonnegative integer matrices with row sums λ_i and column sums μ_j, respectively. Then we have*

$$\mathrm{h}_\lambda = \sum_\mu S_{\lambda\mu} \mathrm{m}_\mu$$

In fact, the transition matrix $S_{\lambda\mu}$ is symmetric.

1.1. **Diagrams.** The homogeneous symmetric functions correspond to single-rowed SSYT's. E.g., h_4 is the sum of all SSYT's with shape

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For $d = 2$, we have

$$\mathrm{h}_3 =$$
$$\boxed{1\,|\,1\,|\,1} + \boxed{1\,|\,1\,|\,2} + \boxed{1\,|\,2\,|\,2} + \boxed{2\,|\,2\,|\,2}$$

the diagrams indicating the corresponding monomials.

Proposition 1.2. *The number of terms in* h_n *is*

$$\#\mathrm{h}_n = \binom{n+d-1}{n}$$

Proof. The number of monomials of degree n in d variables is the coefficient of v^n in the expansion of $(1-v)^{-d}$, namely $\dfrac{(d)_n}{n!}$ which rearranges accordingly. □

Example. For $d = 2$, $n = 3$, we get

$$\#\mathrm{h}_3 = \binom{4}{3} = 4$$

as seen above.