

HOMOGENEOUS SYMMETRIC FUNCTIONS

1. HOMOGENEOUS FAMILY

A homogeneous symmetric function with a single integer index, n , is the sum of all monomials of homogeneous degree n . That is,

$$x_1^{n_1} x_2^{n_2} \cdots x_d^{n_d}$$

has homogeneous degree n if the total degree $n_1 + \cdots + n_d = n$. In terms of monomial symmetric functions,

$$h_n = \sum_{\lambda \vdash n} m_\lambda$$

Example. Let $d = 3$. We have

$$\begin{aligned} h_4 &= x_1^4 + x_2^4 + x_3^4 + x_1^3 x_2 + x_1^3 x_3 + x_1 x_2^3 + x_1 x_3^3 + x_2^3 x_3 + x_2 x_3^3 \\ &\quad + x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2 + x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2 \\ &= m_{(4)} + m_{(31)} + m_{(22)} + m_{(211)} \end{aligned}$$

with $m_{(1111)} = 0$ as $L([1111]) > 3$.

The homogeneous symmetric function indexed by λ is the product of the corresponding single-indexed functions:

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_L} = h_1^{\rho_1} h_2^{\rho_2} \cdots h_n^{\rho_n} = h^\rho$$

in multi-index notation. Taken together $\{h_\lambda\}_\lambda$ form a basis for the symmetric functions under consideration.

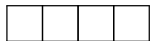
We have the expansion in terms of monomial symmetric functions.

Proposition 1.1. *For given partitions λ, μ , let $S_{\lambda\mu}$ denote the number of nonnegative integer matrices with row sums λ_i and column sums μ_j , respectively. Then we have*

$$h_\lambda = \sum_{\mu} S_{\lambda\mu} m_\mu$$

In fact, the transition matrix $S_{\lambda\mu}$ is symmetric.

1.1. **Diagrams.** The homogeneous symmetric functions correspond to single-rowed SSYT's. E.g., h_4 is the sum of all SSYT's with shape



For $d = 2$, we have

$$h_3 = \boxed{1|1|1} + \boxed{1|1|2} + \boxed{1|2|2} + \boxed{2|2|2}$$

the diagrams indicating the corresponding monomials.

Proposition 1.2. *The number of terms in h_n is*

$$\#h_n = \binom{n+d-1}{n}$$

Proof. The number of monomials of degree n in d variables is the coefficient of v^n in the expansion of $(1-v)^{-d}$, namely $\frac{\binom{d}{n}}{n!}$ which rearranges accordingly. \square

Example. For $d = 2$, $n = 3$, we get

$$\#h_3 = \binom{4}{3} = 4$$

as seen above.