ELEMENTARY SYMMETRIC FUNCTIONS

1. Elementary family

An elementary symmetric function with a single index, n, is the sum of all monomials each consisting of n factors, the variables taken from a subset of size n from the d variables available:

$$\mathrm{e}_n = \sum_{n\text{–subsets of }\{1,\ldots,d\}} x_{i_1} x_{i_2} \cdots x_{i_n}$$

In terms of monomial symmetric functions,

$$\mathrm{e}_n = \mathrm{m}_{(1^n)}$$

In general, $\mathrm{e}_1 = x_1 + \cdots + x_d$, the sum of the x's, and $\mathrm{e}_d = x_1 x_2 \cdots x_d$, their product.

Remark. Note that $\mathrm{e}_n = 0$ if $n > d$.

Example. For example, with $d = 4$, we have

$$\begin{aligned}
\mathrm{e}_1 &= x_1 + x_2 + x_3 + x_4 \\
\mathrm{e}_2 &= x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_3 + x_3x_4 \\
\mathrm{e}_3 &= x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 \\
\mathrm{e}_4 &= x_1x_2x_3x_4
\end{aligned}$$

The elementary symmetric function indexed by λ is the product of the corresponding single-indexed functions:

$$\mathrm{e}_\lambda = \mathrm{e}_{\lambda_1}\mathrm{e}_{\lambda_2} \cdots \mathrm{e}_{\lambda_L} = \mathrm{e}_1^{\rho_1}\mathrm{e}_2^{\rho_2} \cdots \mathrm{e}_n^{\rho_n} = \mathrm{e}^\rho$$

in multi-index notation.
Taken together $\{\mathrm{e}_\lambda\}_\lambda$ form a basis for the symmetric functions under consideration.
We have the expansion in terms of monomial symmetric functions.

Proposition 1.1. *For given partitions λ, μ, let $T_{\lambda\mu}$ denote the number of* 0-1 *matrices with row sums λ_i and column sums μ_j, respectively.*

Then we have

$$\mathrm{e}_\lambda = \sum_\mu T_{\lambda\mu} \mathrm{m}_\mu \ .$$

In fact, the *transition matrix* $T_{\lambda\mu}$ is symmetric.

1.1. **Diagrams.** The elementary symmetric functions correspond to SSYT's consisting of a single column. E.g., e_3 is the sum of all SSYT's with shape

$$\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \end{array}$$

For $d = 4$, we have

$$\mathrm{e}_3 = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} + \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 4 \\ \hline \end{array} + \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 4 \\ \hline \end{array} + \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array}$$

the diagrams indicating the corresponding monomials.

Proposition 1.2. *The number of terms in* e_n *is*

$$\#\mathrm{e}_n = \binom{d}{n}$$

Proof. Each monomial summand is the product of x's with subscripts taken from an n-subset of the d variables. $\square$

Example. For $d = 4$, $n = 3$, we get

$$\#\mathrm{e}_3 = \binom{4}{3} = 4$$

as seen above.