

ELEMENTARY SYMMETRIC FUNCTIONS

1. ELEMENTARY FAMILY

An elementary symmetric function with a single index, n , is the sum of all monomials each consisting of n factors, the variables taken from a subset of size n from the d variables available:

$$e_n = \sum_{n\text{-subsets of } \{1, \dots, d\}} x_{i_1} x_{i_2} \cdots x_{i_n}$$

In terms of monomial symmetric functions,

$$e_n = m_{(1^n)}$$

In general, $e_1 = x_1 + \cdots + x_d$, the sum of the x 's, and $e_d = x_1 x_2 \cdots x_d$, their product.

Remark. Note that $e_n = 0$ if $n > d$.

Example. For example, with $d = 4$, we have

$$\begin{aligned} e_1 &= x_1 + x_2 + x_3 + x_4 \\ e_2 &= x_1 x_2 + x_1 x_3 + x_1 x_4 + x_2 x_3 + x_2 x_4 + x_3 x_4 \\ e_3 &= x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4 \\ e_4 &= x_1 x_2 x_3 x_4 \end{aligned}$$

The elementary symmetric function indexed by λ is the product of the corresponding single-indexed functions:

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_L} = e_1^{\rho_1} e_2^{\rho_2} \cdots e_n^{\rho_n} = e^\rho$$

in multi-index notation.

Taken together $\{e_\lambda\}_\lambda$ form a basis for the symmetric functions under consideration.

We have the expansion in terms of monomial symmetric functions.

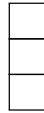
Proposition 1.1. *For given partitions λ , μ , let $T_{\lambda\mu}$ denote the number of 0-1 matrices with row sums λ_i and column sums μ_j , respectively.*

Then we have

$$e_\lambda = \sum_{\mu} T_{\lambda\mu} m_{\mu} .$$

In fact, the *transition matrix* $T_{\lambda\mu}$ is symmetric.

1.1. Diagrams. The elementary symmetric functions correspond to SSYT's consisting of a single column. E.g., e_3 is the sum of all SSYT's with shape



For $d = 4$, we have

$$e_3 =$$

$$\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} + \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 4 \\ \hline \end{array} + \begin{array}{|c|} \hline 1 \\ \hline 3 \\ \hline 4 \\ \hline \end{array} + \begin{array}{|c|} \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array}$$

the diagrams indicating the corresponding monomials.

Proposition 1.2. *The number of terms in e_n is*

$$\#e_n = \binom{d}{n}$$

Proof. Each monomial summand is the product of x 's with subscripts taken from an n -subset of the d variables. \square

Example. For $d = 4$, $n = 3$, we get

$$\#e_3 = \binom{4}{3} = 4$$

as seen above.