GATEWAY

to

SYMMETRIC FUNCTIONS

Crib Notes

Starting with a matrix M
we consider symmetric functions with variables the eigenvalues of M

$d \times d$ Matrix M $\longrightarrow$ $f(x) = \det(xI - M)$

$$f(x) = 0 \Rightarrow x^d = a_1 x^{d-1} + a_2 x^{d-2} + \cdots + a_d$$

Get the coefficients from the characteristic polynomial and construct the following

Recurrence Matrix

$$T = \begin{pmatrix} a_1 & a_2 & \cdots & \cdots & a_d \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

$$\det(xI - T) = \det(xI - M)$$

- **Elementary** Symmetric Functions: $\mathrm{e}_k = (-1)^{k-1} a_k$
- **Homogeneous** Symmetric Functions: $\mathrm{h}_n = (T^n)_{11}$ (top left corner of T^n)
- **Powersum** Symmetric Functions: $\mathrm{p}_n = \mathrm{tr}\, T^n = \mathrm{tr}\, M^n$
- **Monomial** Symmetric Functions: $\det(I + c_1 T + c_2 T^2 + \cdots + c_N T^N) = \sum c_\lambda \mathrm{m}_\lambda$
- **Observe** that $\det(T^n) = \det(T)^n = (\mathrm{e}_d)^n$
- **Note:** $\mathrm{m}_{(n)} = \mathrm{p}_n$ and $\mathrm{m}_{(1^n)} = \mathrm{e}_n$
- **And** $\mathrm{h}_n = \sum_{\lambda \vdash n} \mathrm{m}_\lambda$

Here are some examples for $d = 3$.

$$\begin{aligned}
&\mathrm{e}_1 = x_1 + x_2 + x_3 && \mathrm{h}_1 = x_1 + x_2 + x_3 \\
&\mathrm{e}_2 = x_1x_2 + x_1x_3 + x_2x_3 && \mathrm{h}_2 = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3 \\
&\mathrm{e}_3 = x_1x_2x_3 && \mathrm{h}_3 = x_1^3 + x_2^3 + x_3^3 \\
&\mathrm{e}_n = 0 \quad \text{for } n > 3 && \qquad + x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2x_3^2 + x_1x_2x_3
\end{aligned}$$

$$\begin{aligned}
&\mathrm{p}_1 = x_1 + x_2 + x_3 && \mathrm{m}_{(3)} = x_1^3 + x_2^3 + x_3^3 = \mathrm{p}_3 \\
&\mathrm{p}_2 = x_1^2 + x_2^2 + x_3^2 && \mathrm{m}_{(21)} = x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2x_3^2 \\
&\mathrm{p}_3 = x_1^3 + x_2^3 + x_3^3 && \mathrm{m}_{(111)} = x_1x_2x_3 = \mathrm{e}_3 \\
&\mathrm{p}_n = x_1^n + x_2^n + x_3^n && \mathrm{m}_{(22)} = x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2
\end{aligned}$$

Example. Take $x_1 = 10$, $x_2 = 2$, $x_3 = 1$. We have $\mathrm{e}_1 = 13$, $\mathrm{e}_2 = 32$, $\mathrm{e}_3 = 20$. The recurrence matrix is

$$T = \begin{pmatrix} 13 & -32 & 20 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

with powers of T: T^2, T^3, T^4, ...

$$\begin{pmatrix} 137 & -396 & 260 \\ 13 & -32 & 20 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1385 & -4124 & 2740 \\ 137 & -396 & 260 \\ 13 & -32 & 20 \end{pmatrix}, \begin{pmatrix} 13881 & -41580 & 27700 \\ 1385 & -4124 & 2740 \\ 137 & -396 & 260 \end{pmatrix}, \ldots$$

The sequence $\{\mathrm{h}_n\}_{n\geq 0}$ starts with

$$1,\ 13,\ 137,\ 1385,\ 13881,\ 138873,\ 1388857,\ 13888825,\ \ldots$$

For $\{\mathrm{p}_n\}_{n\geq 1}$ we have the traces

$$13,\ 105,\ 1009,\ 10017,\ 100033,\ 1000065,\ 10000129,\ \ldots$$

with $\det(T^n) = \det(T)^n = 20^n$. For monomial functions, we can expand, taking indices up to length 3,

$$\begin{aligned}
&\det(I + c_1T + c_2T^2 + c_3T^3) \\
&\ = 1 + 20c_1^3 + 260c_1^2c_2 + 2100c_1^2c_3 + 32c_1^2 \\
&\ + 640c_1c_2^2 + 7120c_1c_2c_3 + 356c_1c_2 + 10080c_1c_3^2 + 3100c_1c_3 + 13c_1 \\
&\ + 400c_2^3 + 5200c_2^2c_3 + 504c_2^2 + 12800c_2c_3^2 + 5912c_2c_3 + 105c_2 + 8000c_3^3 + 9008c_3^2 + 1009c_3
\end{aligned}$$

with, e.g., taking coefficients of $c_{\lambda_1}c_{\lambda_2}c_{\lambda_3}$, we have $\mathrm{m}_{(1)} = \mathrm{e}_1 = 13$,

$$\mathrm{m}_{(11)} = \mathrm{e}_2 = 32,\ \mathrm{m}_{(111)} = \mathrm{e}_3 = 20,\ \mathrm{m}_{(3)} = \mathrm{p}_3 = 1009,\ \mathrm{m}_{(21)} = 356, \text{ and } \mathrm{m}_{(332)} = 12800$$

and $\mathrm{h}_3 = \mathrm{m}_{(3)} + \mathrm{m}_{(21)} + \mathrm{m}_{(111)} = 1009 + 356 + 20 = 1385$ checks.

S-functions from T

For a partition $\lambda = [\lambda_1\lambda_2\cdots]$, the S-function $\{\lambda\}$ is defined as the ratio of determinants in the variables $\{x_1, x_2 \ldots, x_d\}$

$$\boxed{\{\lambda\} = \det(x_j^{\lambda_i+d-i})/\det(x_j^{d-i})}$$

where $\det(x_j^{d-i})$ is the *Vandermonde determinant.*

$$\boxed{\text{Define the matrix entries: } \xi_n^{(j)} = (T^n)[1,j]}$$

i.e., $\xi_n^{(j)}$ is the j^{th} entry in the top row of $T^n = \left(\xi_{n-i+1}^{(j)}\right)_{1\le i,j\le d}$.

So

$$\xi_1^{(j)} = a_j \quad \text{and} \quad \xi_n^{(1)} = \mathrm{h}_n$$

and values for $n < 0$ are determined by the relation $T^0 = I$.

Then

$$\boxed{\{\lambda\} = \det(\xi_{\lambda_i-i+1}^{(j)})}$$

Example. Let us find $\{21\}$ for $d = 3$. From the definition directly we have

$$\{21\} = \frac{\begin{vmatrix} x_1^4 & x_2^4 & x_3^4 \\ x_1^2 & x_2^2 & x_3^2 \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} x_1^2 & x_2^2 & x_3^2 \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix}} = \frac{x_1^4x_2^2 - x_1^4x_3^2 - x_1^2x_2^4 + x_1^2x_3^4 + x_2^4x_3^2 - x_2^2x_3^4}{x_1^2x_2 - x_1^2x_3 - x_1x_2^2 + x_1x_3^2 + x_2^2x_3 - x_2x_3^2}$$

Factoring, we have

$$\begin{aligned} \{21\} &= \frac{(x_1-x_2)(x_1+x_2)(x_1-x_3)(x_1+x_3)(x_2-x_3)(x_2+x_3)}{(x_1-x_2)(x_1-x_3)(x_2-x_3)} \\ &= (x_1+x_2)(x_1+x_3)(x_2+x_3) \\ &= x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + 2x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 \\ &= \mathrm{m}_{(21)} + 2\,\mathrm{m}_{(111)} \end{aligned}$$

In our numerical example, we compare with

$$\begin{vmatrix} \xi_2^{(1)} & \xi_2^{(2)} \\ \xi_0^{(1)} & \xi_0^{(2)} \end{vmatrix} = \begin{vmatrix} 137 & -396 \\ 1 & 0 \end{vmatrix} = 396$$

using the values from T^2 and T^0. And substituting the values $x_1 = 10$, $x_2 = 2$, $x_3 = 1$ yields 396 accordingly.

Similarly, we find

$$\{22\} = x_1^2x_2^2 + x_1^2x_2x_3 + x_1^2x_3^2 + x_1x_2^2x_3 + x_1x_2x_3^2 + x_2^2x_3^2 = \mathrm{m}_{(22)} + \mathrm{m}_{(211)}$$

compared with

$$\begin{vmatrix} \xi_2^{(1)} & \xi_2^{(2)} \\ \xi_1^{(1)} & \xi_1^{(2)} \end{vmatrix} = \begin{vmatrix} 137 & -396 \\ 13 & -32 \end{vmatrix} = 764$$