Symmetric Powers Fibonacci Example

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April 12, 2016

In [1]:
```
from IPython.display import *
```

In [29]:
```
var('a:z')
V=Matrix(2,1,[u,v])
A=Matrix(2,2,[1,1,1,0])
B=A*V
display(Math("A="),A)
```

$$A =$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

In [3]:
```
N=5
mm=Poly((u+v)**N).monoms()
```

In [4]:
```
dd=len(mm)
xx=eye(dd)
X=[]

for i in range(dd):
    F=x**mm[i][0]*y**mm[i][1]
    GG=F.subs(x,B[0]).subs(y,B[1]);FF=expand(GG)
    X.append([FF.coeff(u**mm[i][0]*v**mm[i][1]) for i in range(dd)])

#display(X)
```

In [5]:
```
MX=Matrix(1,dd,X[0])
for i in range(1,dd):
    XM=Matrix(1,dd,X[i])
    MX=MX.col_join(XM)


MX
```

Out[5]:

$$\begin{bmatrix} 1 & 5 & 10 & 10 & 5 & 1 \\ 1 & 4 & 6 & 4 & 1 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix with binomial coefficients is the symmetric tensor power of the Fibonacci matrix A.

```
In [20]: IX=eye(A.shape[0])
         delta=series(((IX-t*A).det())**(-1),t,0,10)
         for i in range(10):
             display(Math("F_{"+str(i+1)+ "} = "+str(delta.coeff(t,i))))
```

$$F_1 = 1$$

$$F_2 = 1$$

$$F_3 = 2$$

$$F_4 = 3$$

$$F_5 = 5$$

$$F_6 = 8$$

$$F_7 = 13$$

$$F_8 = 21$$

$$F_9 = 34$$

$$F_{10} = 55$$

The Fibonacci numbers are the traces of the matrices formed from the binomial coefficients.

```
In [7]: MXX=MX.eigenvals()
        MXE=list(MXX.keys())
```

```
In [8]: var('phi')
        phi=GoldenRatio
```

```
In [9]: [nsimplify(MXE[i],[GoldenRatio]) for i in range(len(MXE))]
```

Out[9]:

$$[-5\phi + 8, \quad 3 + 5\phi, \quad -\phi + 1, \quad \phi, \quad -3 + 2\phi, \quad -2\phi - 1]$$

```
In [10]: [(phi**(N-i)*(-phi)**(-i)) for i in range(N+1)]
```

Out[10]:

$$\left[\phi^5, \quad -\phi^3, \quad \phi, \quad -\frac{1}{\phi}, \quad \frac{1}{\phi^3}, \quad -\frac{1}{\phi^5}\right]$$

```
In [11]: [nsimplify(phi**(N-i)*(-phi)**(-i),[GoldenRatio]) for i in range(N+1)]
```

Out[11]:

$$[3+5\phi, \quad -2\phi-1, \quad \phi, \quad -\phi+1, \quad -3+2\phi, \quad -5\phi+8]$$

The eigenvalues are the monomials $(\phi)^{N-i}(-\phi)^{-i}$ in the eigenvalues of A.

In [19]:
```
Res=series(trace((IX-t*A).inv()),t,0,10)
for i in range(10):
    display(Math("L_{"+str(i)+ "} = "+str(Res.coeff(t,i))))
```

$$L_0 = 2$$

$$L_1 = 1$$

$$L_2 = 3$$

$$L_3 = 4$$

$$L_4 = 7$$

$$L_5 = 11$$

$$L_6 = 18$$

$$L_7 = 29$$

$$L_8 = 47$$

$$L_9 = 76$$

This is the beginning of the sequence of Lucas numbers. Traces of the powers of A.